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ON LIOUVILLE'S NORMAL FORM FOR LANCHESTER-TYPE EQUATIONS OF MODERN WARFARE WITH VARIABLE COEFFICIENTS

by

James G. Taylor

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ON LIOUVILLE'S NORMAL FORM FOR LANCHESTER-TYPE EQUATIONS OF MODERN WARFARE WITH VARIABLE COEFFICIENTS

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ABSTRACT

This paper shows that much new information about the dynamics of combat between two homogeneous forces modelled by Lanchester-type equations of modern warfare (also frequently referred to as "square-law" attrition equations) with temporal variations in fire effectivenesses (as expressed by the Lanchester attrition-rate coefficients) may be obtained by considering Liouville's normal form for the X and Y force-level equations. It is shown that the relative fire effectiveness of the two combatants and the intensity of combat are two key parameters determining the course of such Lanchester-type combat. New victory-prediction conditions that allow one to forecast the battle's outcome without explicitly solving the deterministic combat equations and computing force-level trajectories are developed for fixed-force-ratio-breakpoint battles by considering Liouville's normal form. These general results are applied to two special cases of combat modelled with general power attrition-rate coefficients. A refinement of a previously known victory-prediction condition is given. Temporal variations in relative fire effectiveness play a central role in these victory-prediction results. Liouville's normal form is also shown to yield an approximation to the force-level trajectories in terms of elementary functions.

1. Introduction.

Even though combat between two military forces is a complex random process (see Note 1 of TAYLOR and BROWN[50]), as a consequence of the pioneering 1914 work of F. W. LANCHESTER[26] (see Note 1) military operations analysts since about the end of World War II have used simplified deterministic differential equation models to develop insights into the dynamics of combat (see Note 2). Today Lanchester-type models of quite complex military systems have been developed in the United States (see, for example, BONDER and HONIG[9]) and require a digital computer for their implementation. A simple combat model, however, may yield an understanding of important relations that are difficult to perceive in a more complex model, and such insights may provide guidance for higher resolution computerized investigations (see, for example, BONDER and FARREL[8] and WEISS[55]). In this paper we will examine such an idealized Lanchester-type model in order to obtain some insights (specifically, the tradeoff between quality and quantity of weapon systems) into the dynamics of combat between two homogeneous forces with temporal variations in weapon system effectivenesses.

In this paper we develop new victory-prediction conditions that sometimes allow us to forecast the battle's outcome without explicitly solving the combat equations (see Note 3) and computing force-level trajectories (see Note 4). We obtain these results for variable-coefficient Lanchester-type equations of modern warfare by considering Liouville's normal form of the X and Y force-level equations (see p. 270 of INCE[23] and KAMKE[24]) and using techniques recently applied to Lanchester combat theory by TAYLOR and PARRY[51]. These results complement and extend those of Taylor and Parry[51], and they show that the key parameters affecting a battle's outcome, at least for fixed-force-ratio-breakpoint battles with the initial force ratio held constant, are the relative effectiveness of the weapon systems and the intensity of combat. Such results are not only important in their own right but also useful in the quantitative analysis of tactics (see, for example, TAYLOR[44-46]).

This paper is organized in the following fashion. First, we review F. W. Lanchester's classic mathematical model of combat between two homogeneous forces and its extension to cases of time-varying fire effectivenesses. Next, we transform first the independent variable (time, t) and then the dependent variable (force level) to obtain Liouville's normal form for the X and Y force-level equations. Then we develop some new victory-prediction conditions from Liouville's normal form and apply these general results to two special cases of power attrition-rate coefficients. Finally, we discuss the significance of our developments.

2. Lanchester's Classic Formulation.

F. W. Lanchester[26] hypothesized (see Note 5) in 1914 (see Note 6) that combat between two military forces "under modern conditions" could be modelled by (see Note 7)

$$\begin{cases} dx/dt = -ay & \text{with} \quad x(t=0) = x_0, \\ dy/dt = -bx & \text{with} \quad y(t=0) = y_0, \end{cases} \tag{1}$$

where $t = 0$ denotes the time at which the battle begins, $x(t)$ and $y(t)$ denote the numbers of X and Y at time t, and a and b are nonnegative constants that are today called Lanchester attrition-rate coefficients and represent each side's fire effectiveness. Lanchester considered this simple model in order to provide insights into the dynamics of combat under "modern conditions" and justify the principle of concentration (see Note 8). We will accordingly refer to (1) as Lanchester's equations of modern warfare. Various sets of physical circumstances have been hypothesized to yield them: for example, (A) both sides use aimed fire and target acquisition times are constant (see WEISS[54]), or (B) both sides use area fire and a constant density defense (see BRACKNEY[11]). Other forms of Lanchester-type equations have appeared in the literature, but we will not consider these here (see DOLANSKY[15] and TAYLOR[43, 48]).

From (1) Lanchester deduced his famous square law

$$b\{x_0^2 - x^2(t)\} = a\{y_0^2 - y^2(t)\}, \tag{2}$$

which has the important implication that a side can significantly reduce its casualties by initially committing more forces to battle. It follows from (2) that

$$X \text{ will be annihilated} \Leftrightarrow \frac{x_0}{y_0} < \sqrt{\frac{a}{b}} . \tag{3}$$

Unfortunately, no simple relationship similar to (2) holds in general for variable attrition-rate coefficients so we consider other means for developing (3). As is well known, the X force-level, $x(t)$, is given by

$$x(t) = x_0 \cosh(\sqrt{ab}\, t) - y_0 \sqrt{\frac{a}{b}} \sinh(\sqrt{ab}\, t). \tag{4}$$

We may also deduce (3) by writing (4) as

$$x(t) = \frac{1}{2}\{(x_0 - y_0\sqrt{\frac{a}{b}})\exp(\sqrt{ab}\, t) + (x_0 + y_0\sqrt{\frac{a}{b}})\exp(-\sqrt{ab}\, t)\} , \tag{5}$$

and observing that $x(t)$ can become zero if and only if the coefficient of the increasing exponential is negative. We observe from (5) that annihilation occurs in finite time.

As H. K. WEISS[53] has emphasized, engagements that continue until one side is wiped out are rare. Thus, we see that a model of battle termination is required. Although we are well aware that battle termination is a complex random process for which it is by no means certain that force levels are the only significant variables (see Note 9), we assume that combat ends when either of two given "breakpoint" force ratios is reached. Introducing the force ratio $u = x/y$, we have that these "breakpoint" force ratios, denoted as u_X^f when X wins and u_Y^f when Y wins, satisfy $0 \leq u_Y^f < u_0 = u(t=0) < u_X^f \leq +\infty$. Corresponding to a fight until the annihilation of one side or the other is the case in which $u_Y^f = 0$ and $u_X^f = +\infty$.

Let us now consider such a fixed-force-ratio-breakpoint battle. Introducing the force ratio, $u = x/y$, we see that it satisfies the Riccati equation

$$du/dt = bu^2 - a \qquad \text{with} \quad u(t=0) = u_0 = x_0/y_0. \tag{6}$$

Let $u_+ = \sqrt{a/b}$ denote the positive root of the quadratic equation $bu^2 - a = 0$ and observe that $du/dt < 0 \Leftrightarrow u < u_+$ (see Figure 2 of Taylor and Parry[51]). In

particular, $u_0 < u_+ \Rightarrow du/dt(t) \leq du/dt(t=0) < 0$, whence follows

THEOREM 1: Consider Lanchester's equations of modern warfare with constant attrition-rate coefficients (1). Then X will lose a fixed-force-ratio breakpoint battle in finite time if and only if $x_0/y_0 < \sqrt{a/b}$.

We observe that (3) is a special case of Theorem 1 corresponding to $u_Y^f = 0$ and $u_X^f = +\infty$. One result of this paper is to generalize [in a way different from that given by Taylor and Parry[51] (see Section 6)] Theorem 1 to cases of variable attrition-rate coefficients.

3. Variable Attrition-Rate Coefficients.

The pioneering work of S. BONDER[3, 5, 8] on methodology for the evaluation of military systems (especially mobile systems such as tanks, mechanized infantry combat vehicles, etc.) has generated interest in variable-coefficient Lanchester-type equations and has led to improved operations research techniques for the prediction of these coefficients (see Note 10). Let us therefore consider

$$\begin{cases} dx/dt = -a(t)y & \text{with} \quad x(t=0) = x_0, \\ dy/dt = -b(t)x & \text{with} \quad y(t=0) = y_0, \end{cases} \tag{7}$$

where $a(t)$ and $b(t)$ denote time-dependent Lanchester attrition-rate coefficients. These coefficients depend on such variables as force separation, tactical posture of targets, rate of targer acquisition, firing doctrine, firing rate, etc. (see reference 8). We will also refer to (7) as the equations for a square-law attrition process, since an "instantaneous" square law holds even when $a(t)/b(t)$ is not constant (see Taylor and Parry[51]; also references 45 and 47).

A large class of tactical situations of interest can be modelled with the following general power attrition-rate coefficients (see reference 8)

$$a(t) = k_a(t+C)^{\mu}, \qquad \text{and} \qquad b(t) = k_b(t+C+A)^{\nu}, \tag{8}$$

where $A,C \geq 0$. We will call A the offset parameter, since it allows us to model (with $\mu,\nu \geq 0$) battles between weapon systems with different effective ranges (see Note 11). We will call C the starting parameter, since it allows us to model (again, with $\mu,\nu \geq 0$) battles that begin within the maximum effective ranges of the two systems. Restrictions that must be placed on μ and ν, which are not necessarily integers, are discussed below.

The above nomenclature is motivated and possible applications of our results are indicated by considering S. Bonder's[3, 5] model of the constant-speed attack on a static defensive position

$$dx/dt = -\alpha(r)y = -\alpha_0(1-r/R_\alpha)^\mu y, \qquad dy/dt = -\beta(r)x = -\beta_0(1-r/R_\beta)^\nu x, \tag{9}$$

where r denotes the range between opposing forces, $\mu,\nu \geq 0$, and R_α denotes the maximum effective range of the Y weapon system. Range is related to time by $r(t) = R_0 - vt$, where R_0 denotes the opening range of battle and $v > 0$ denotes the constant attack speed. Then the offset and starting parameters are given by

$$A = (R_\beta - R_\alpha)/v, \qquad \text{and} \qquad C = (R_\alpha - R_0)/v. \tag{10}$$

We observe that $A,C \geq 0$ if and only if $R_\beta \geq R_\alpha \geq R_0$. By considering (10) and Figure 1, the reader should have no trouble in understanding our terminology for A and C. In the model (9) μ, for example, is used to model the range dependence of Y's attrition-rate coefficient (see Figure 2).

From (7) we obtain the X force-level equation

$$\frac{d^2x}{dt^2} - \{\frac{d}{dt}\ln a(t)\}\frac{dx}{dt} - a(t)b(t)x = 0, \tag{11}$$

with initial conditions

$$x(t=0) = x_0 \quad \text{and} \quad \{[1/a(t)]dx/dt\}_{t=0} = -y_0,$$

where $t_0 = \max(t_0^X, t_0^Y)$, and t_0^X denotes the right most finite singularity of the X force-level equation (see Ince[23]). The coefficients $a(t)$ and $b(t)$ are then positive continuous functions $\forall\, t > t_0$. For the coefficient (8), we observe that

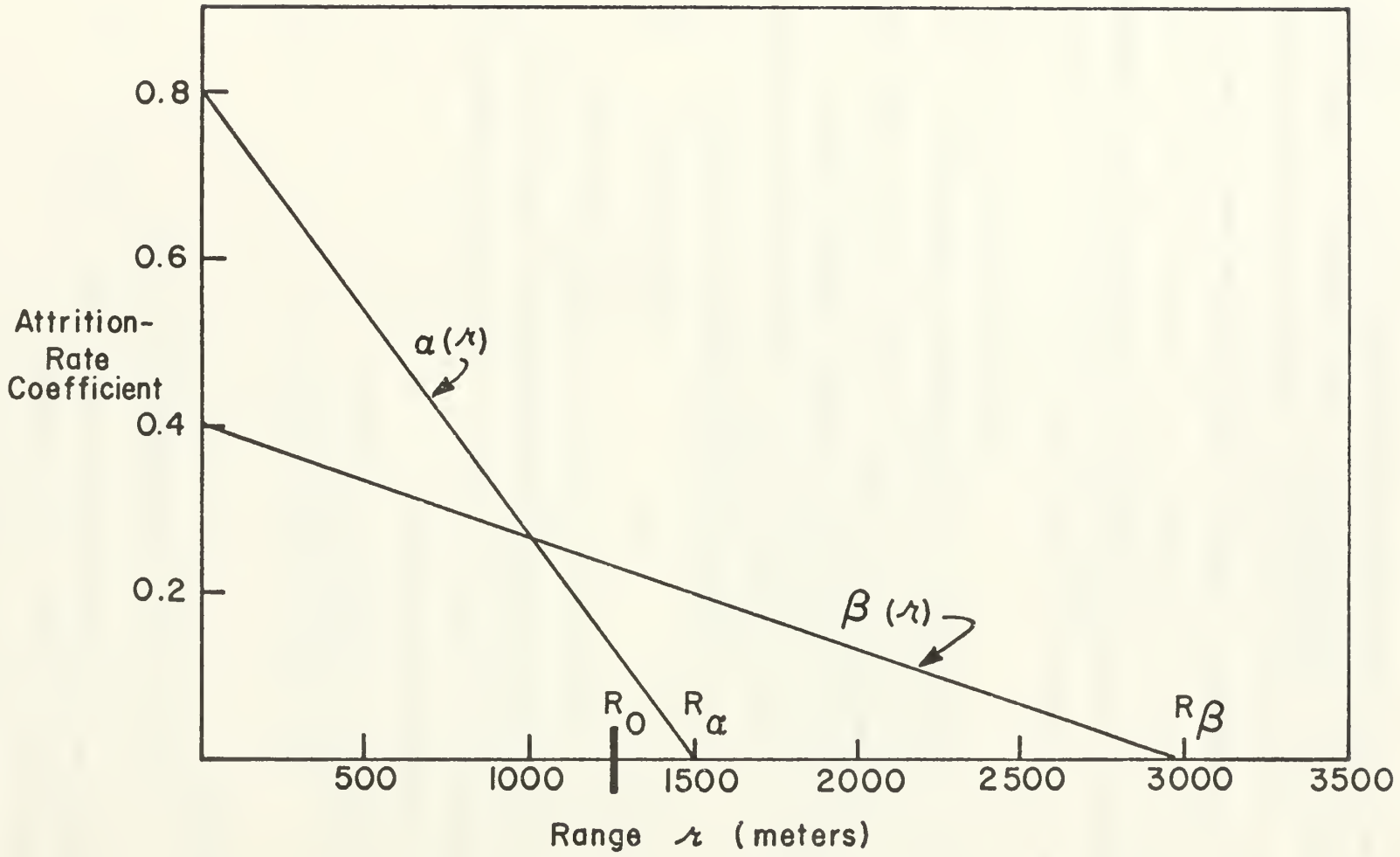

Figure 1. Explanation of offset parameter A and starting parameter C for power attrition-rate coefficients modelling constant-speed attack. [Notes: 1. The maximum effective ranges of the two weapon systems are denoted as R_α and R_β. 2. The opening range of battle is denoted as R_0 and (as shown) R_0 < minimum (R_α, R_β). 3. The offset parameter is given by $A = (R_\beta - R_\alpha)/v$. 4. The starting parameter is given by $C = (R_\alpha - R_0)/v$.]

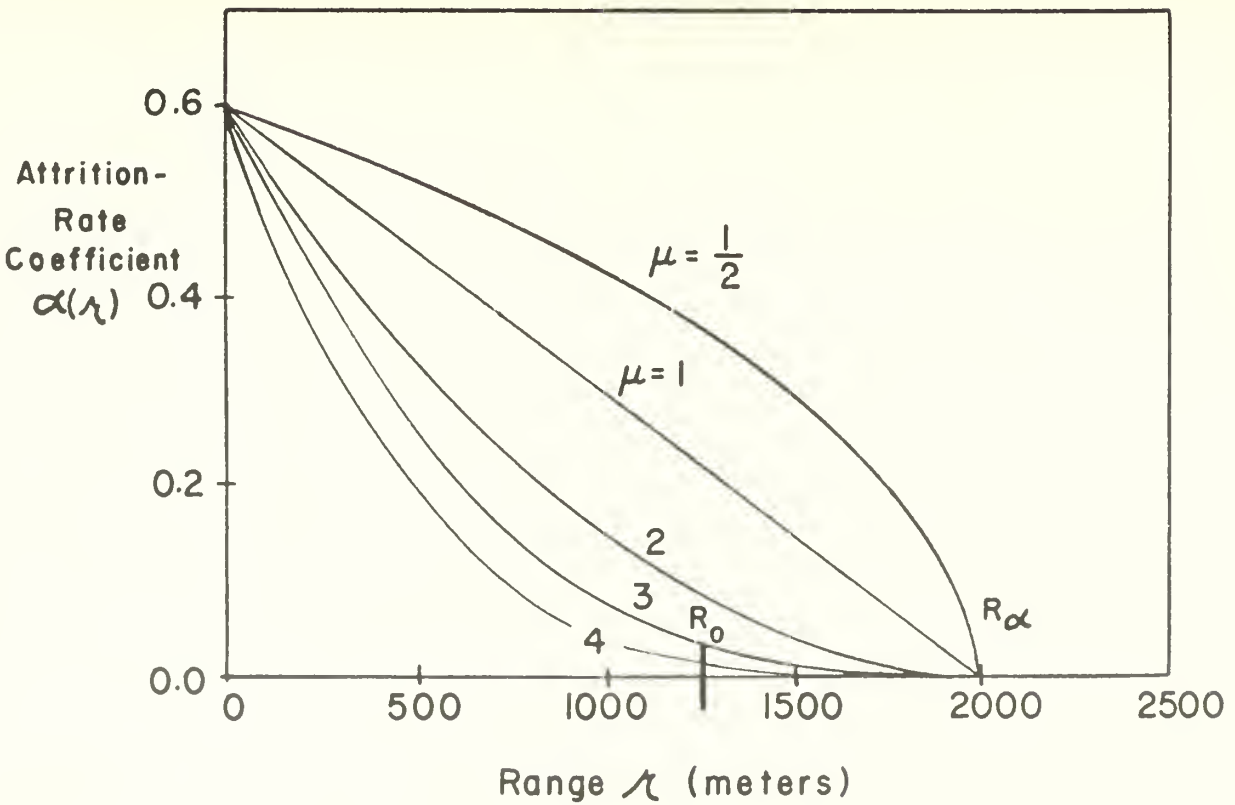

Figure 2. Dependence of the attrition-rate coefficient $\alpha(r)$ on the exponent μ with maximum effective range of the weapon system and kill capability at zero range held constant. [Notes: 1. The maximum effective range of the system is denoted as $R_\alpha = 2000$ meters. 2. $\alpha(r=0) = \alpha_0 = 0.6$ X casualties/(unit time × number of Y units) denotes the Y force weapon system kill rate at zero force separation (range). 3. The opening range of battle is denoted as $R_0 = 1250$ meters and (as shown) $R_0 < R_\alpha$.]

$t_0 = -C$. Moreover, to insure the existence of all derivatives required in subsequent analysis, we assume that the second derivatives of $a(t)$ and $b(t)$ exist $\forall\, t > t_0$. We similarly have the Y force-level equation

$$\frac{d^2y}{dt^2} - \{\frac{d}{dt} \ell n\, b(t)\}\frac{dy}{dt} - a(t)b(t)y = 0, \tag{12}$$

with initial conditions

$$y(t=0) = y_0 \qquad \text{and} \qquad \{[1/b(t)]dy/dt\}_{t=0} = -x_0.$$

It is necessary to place further restrictions on $a(t)$ and $b(t)$ in order to insure that, for example, the transformation introduced in the next section is well defined. Thus, we assume that the following condition holds.

CONDITION (A): $\int_{t_0}^{t} a(s)ds$ and $\int_{t_0}^{t} b(s)ds$ are bounded for all finite $t \geq t_0$.

This condition also guarantees (see Theorem 6.4.2 on p. 226 of HILLE[22]) that (7) has a continuous solution for all finite $t \geq 0 \geq t_0$ (see Note 12). If Condition (A) is to hold, then for the general power attrition-rate coefficients we must have $\mu,\nu > -1$.

4. Transformation of the Battle's Time Scale.

Let

$$\tau = \int_{t_0}^{t} \sqrt{a(s)b(s)}\, ds\,, \tag{13}$$

and denote $\tau(t=0)$ as τ_0. It follows that $\tau_0 \geq 0$ for $t_0 \leq 0$. Condition (A) guarantees that $\tau = \tau(t)$ is well defined (i.e. bounded for all finite $t \geq t_0$) by the Cauchy-Schwarz inequality for integrals (see p. 123 of BELLMAN[2]). The transformation is invertible, since $d\tau/dt = \sqrt{a(t)b(t)} > 0$ for $t > t_0$. Thus, we may consider that $t = t(\tau)$. We assume (and give below conditions that guarantee) that $\lim_{t\to+\infty} \tau(t) = +\infty$ so that the range of $\tau(t)$ is $[0,+\infty)$ for $t \in [t_0,+\infty)$.

Considering the constant coefficient result (4), we will call the quantity $\sqrt{a(t)b(t)}$ the "intensity of combat" (see also Taylor and Parry[51]); since the larger

it is, the more quickly the battle is moving towards termination. Then $\tau - \tau_0$ is related to the average intensity of combat by

$$\tau - \tau_0 = \{(1/t) \int_0^t \sqrt{a(s)b(s)}\, ds\}t = \sqrt{\overline{a(t)b(t)}}\, t. \tag{14}$$

Applying the transformation (13) to (11) and (12), we obtain

$$\frac{d^2x}{d\tau^2} + \{\frac{1}{2} \frac{d}{d\tau} \ell n \left[\frac{b(t)}{a(t)}\right]\} \frac{dx}{d\tau} - x = 0, \tag{15}$$

with initial conditions

$$x(\tau=\tau_0) = x_0 \quad \text{and} \quad \{[b(t)/a(t)]^{1/2} dx/d\tau\}_{\tau=\tau_0} = -y_0,$$

and

$$\frac{d^2y}{d\tau^2} + \{\frac{1}{2} \frac{d}{d\tau} \ell n \left[\frac{a(t)}{b(t)}\right]\} \frac{dy}{d\tau} - y = 0, \tag{16}$$

with initial conditions

$$y(\tau=\tau_0) = y_0 \quad \text{and} \quad \{[a(t)/b(t)]^{1/2} dy/d\tau\}_{\tau=\tau_0} = -x_0.$$

Taylor and Brown[50] have shown that the X force-level equation (11) may be transformed into a linear second order differential equation with constant coefficients if and only if

$$\frac{1}{\sqrt{a(t)b(t)}} \frac{d}{dt} \ell n \left[\frac{a(t)}{b(t)}\right] = \text{CONSTANT}, \tag{17}$$

with the desired transformation being given by $\tau = K \int^t \sqrt{a(s)b(s)}\, ds$, where $\int^t \ldots ds$ denotes an indefinite integral and K an arbitrary constant. Hence, equation (11) may be transformed into such a constant coefficient equation if and only if (15) is a constant coefficient equation. We observe that (15) has a solution in terms of elementary transcendental functions in the quasi-autonomous case in which the ratio of attrition-rate coefficients is constant (see Note 13), i.e.

$$b(t)/a(t) = \text{CONSTANT}. \tag{18}$$

Except when (17) holds, the solution to (11) is complex, and the qualitative behavior of force-level trajectories has been difficult to establish. In particular, one is interested in answering such questions as

(Q1) Who will "win"? Be annihilated?

(Q2) How do force levels decrease over time and how many survivors will the winner have?

(Q3) How do changes in the initial force levels and/or weapon system parameters affect the outcome? Is concentration of forces a good tactic?

(Q4) How long will the battle last?

We will now show how to answer question (Q1) without explicitly solving the Lanchester-type equations (7).

Equation (15) is highly significant because it clearly shows that the course of combat depends on just the two weapon system parameters: (1) $R(t) = a(t)/b(t)$, the relative effectiveness (Y to X) of the two weapon systems, and (2) $I(t) = \sqrt{a(t)b(t)}$, the intensity of combat (through equation (13), which relates $I(t)$ to τ). Both these parameters may vary over time, and equation (15) tells us that the nature of such temporal variations in relative effectiveness has a significant effect upon the course of combat. Moreover, this relationship may be more explicitly seen by transforming (15) to Liouville's normal form.

5. Reduction to Liouville's Normal Form.

Let us assume that $\tau_0 > 0$ (i.e. $0 > t_0$) so that $a(t)/b(t)$ is twice differentiable and satisfies $0 < a(t)/b(t) < +\infty$ for $0 \leq t < +\infty$. Let a_0 denote $a(t=0)$, etc., and recall that $t = t(\tau)$. The substitution

$$x(\tau) = X(\tau)\left[\frac{a(t)/a_0}{b(t)/b_0}\right]^{1/4} \tag{19}$$

transforms (15) into the so-called normal form (see RAINVILLE[36]) with the first derivative of the dependent variable removed

$$\frac{d^2X}{d\tau^2} - \{1 + F(\tau)\}X = 0, \tag{20}$$

with initial conditions

$$X(\tau=\tau_0) = x_0, \quad \text{and} \quad dX/d\tau(\tau=\tau_0) = -y_0\sqrt{a_0/b_0} - x_0\varepsilon_0,$$

where

$$F(\tau) = P''(\tau)/P(\tau), \qquad P(\tau) = [R(t)]^{-1/4}, \tag{21}$$

$$R(t) = a(t)/b(t), \qquad \varepsilon(t) = \frac{1}{4\sqrt{a(t)b(t)}} \frac{d}{dt} \ell n\, R, \tag{22}$$

ε_0 denotes $\varepsilon(t=0)$, and $P'(\tau)$ denotes $dP/d\tau$. Equation (20) is Liouville's celebrated normal form (see p. 23 of LIOUVILLE[28]). Similarly,

$$y(\tau) = Y(\tau)\left[\frac{b(t)/b_0}{a(t)/a_0}\right]^{1/4} \tag{23}$$

yields

$$\frac{d^2Y}{d\tau^2} - \{1 + G(\tau)\}Y = 0, \tag{24}$$

with initial conditions

$$Y(\tau=\tau_0) = y_0, \quad \text{and} \quad dY/d\tau(\tau=\tau_0) = -x_0\sqrt{b_0/a_0} + y_0\varepsilon_0,$$

where

$$G(\tau) = Q''(\tau)/Q(\tau), \quad \text{and} \quad Q(\tau) = [R(t)]^{1/4} = 1/P(\tau). \tag{25}$$

We observe that

$$F(\tau) + G(\tau) = 2(\frac{d}{d\tau} \ell n\, P)^2. \tag{26}$$

It is sometimes convenient to express F and G in terms of the old time variable t. Then

$$F(\tau) = \frac{1}{4b^2(t)} \frac{d}{dt} \{-\frac{d}{dt} \ell n\, b(t) - \frac{1}{4} \frac{d}{dt} \ell n\, R(t) + \frac{d}{dt} \ell n \frac{dR}{dt}\}, \tag{27}$$

and

$$G(\tau) = \frac{1}{8a(t)b(t)} \{\frac{d}{dt} \ell n\, R(t)\}^2 - F(\tau). \tag{28}$$

Observing that (21) may also be written as

$$F(\tau) = \frac{d^2}{d\tau^2} \ell n\, P(\tau) + \{\frac{d}{d\tau} \ell n\, P(\tau)\}^2, \tag{29}$$

we see that $d\ell n\, P/d\tau = \text{constant}$ (or, equivalently, $\varepsilon(t) = \text{constant}$) implies that $F(\tau)$ is constant, although the converse may not be true.

Writing (20) as $d^2X/d\tau^2 - X = F(\tau)X$, we may use variation of parameters (see pp. 122-123 of Ince[23]) to obtain the solution to (20) as

$$X(\tau) = x_0 \cosh(\tau-\tau_0) - [y_0\sqrt{a_0/b_0} + x_0\varepsilon_0]\sinh(\tau-\tau_0) + \int_{\tau_0}^{\tau} F(\sigma)\sinh(\tau-\sigma)X(\sigma)d\sigma. \quad (30)$$

In terms of the original time variable t and dependent variable x, we have

$$x(t) = \left[\frac{a(t)/a_0}{b(t)/b_0}\right]^{1/4}\{x_0\cosh\left(\int_0^t \sqrt{a(s)b(s)}\,ds\right) - [y_0\sqrt{\frac{a_0}{b_0}} + x_0\varepsilon_0]\sinh\left(\int_0^t \sqrt{a(s)b(s)}\,ds\right)\}$$

$$+ \left[\frac{a(t)}{b(t)}\right]^{1/4}\int_0^t \frac{d}{ds}\{\frac{1}{a(s)b(s)}\frac{d}{ds}\left[\frac{b(s)}{a(s)}\right]^{1/4}\}\sinh\left(\int_s^t \sqrt{a(\sigma)b(\sigma)}\,d\sigma\right)x(s)ds. \quad (31)$$

The value of the Volterra integral equation (30), however, is not so much for direct computation as it is for suggesting an approximation, the so-called Liouville-Green approximation (see OLVER[32]), to the solution of the X force-level equation (11). If the appropriate fractional power of the relative effectiveness is "slowly varying," then by (21) we would expect that $|F(\tau)| << 1$ so that we could drop the integral term in (30) to obtain

$$\hat{X}(\tau) = x_0\cosh(\tau-\tau_0) - [y_0\sqrt{a_0/b_0} + x_0\varepsilon_0]\sinh(\tau-\tau_0), \quad (32)$$

where $\hat{X}(\tau)$ denotes the Liouville-Green approximation. A theoretical error analysis appears in Olver[32] (see also OLVER[33]), although we will not pursue this matter further here. We observe that $F(\tau) \geq 0\ \forall\ \tau \geq \tau_0$ implies that while $X(\tau) \geq 0$ we have $X(\tau) \geq \hat{X}(\tau)$ and similarly when $F(\tau) \leq 0$. As we shall see below, such cases in which $F(\tau)$ is always ≥ 0 or ≤ 0 are readily encountered in applications. In terms of the original variables t and x, we have

$$\hat{x}(t) = \left[\frac{a(t)/a_0}{b(t)/b_0}\right]^{\frac{1}{4}}\{x_0\cosh\left(\int_0^t \sqrt{a(s)b(s)}\,ds\right) - [y_0\sqrt{\frac{a_0}{b_0}} + x_0\varepsilon_0]\sinh\left(\int_0^t \sqrt{a(s)b(s)}\,ds\right)\}, \quad (33)$$

which we may also write in terms of the average combat intensity, $\overline{\sqrt{a(t)b(t)}} = (1/t)\int_0^t \sqrt{a(s)b(s)}\,ds$, as

$$\hat{x}(t) = \left[\frac{a(t)/a_0}{b(t)/b_0}\right]^{\frac{1}{4}}\{x_0 \cosh(\overline{\sqrt{a(t)b(t)}}\, t) - [y_0\sqrt{\frac{a_0}{b_0}} + x_0\varepsilon_0]\sinh(\overline{\sqrt{a(t)b(t)}}\, t)\}. \qquad (34)$$

6. Some New Victory-Prediction Conditions.

In this section we show that much valuable information about the course of combat (for example, force-annihilation prediction) may be obtained directly from Liouville's normal form (20) without making any kind of approximation. Let us assume that the model (7) holds for all time and consider a fixed-force-ratio-breakpoint battle (see Section 2). In order to insure that the battle terminates in finite time, we must make certain technical assumptions about the mathematical nature of the attrition-rate coefficients in (7). For the reader's convenience, we list in Table I the principal such conditions that we use in subsequent developments. We observe that all these conditions hold for constant attrition-rate coefficients. Let us also observe that Conditions (B) and (ND) imply Condition (C).

We record here for future reference Taylor and Parry's[51] generalization of Theorem 1 to cases of variable attrition-rate coefficients (see Note 14).

THEOREM 2: Assume that Conditions (B) and (ND) hold. Then $x_0/y_0 < \sqrt{a_0/b_0}$ implies that the X force will lose a fixed-force-ratio-breakpoint battle in finite time.

PROOF: Introducing the force ratio, $u = x/y$, we have

$$du/dt = b(t)u^2 - a(t) \qquad \text{with} \quad u(t=0) = u_0 = x_0/y_0. \qquad (35)$$

Let $u_+(t) = \sqrt{a(t)/b(t)} = \sqrt{R(t)}$ denote the positive root of the quadratic equation $b(t)u^2 - a(t) = 0$, and observe that $du/dt < 0$ for any $u < u_+(t)$ (see Figure 2 of Taylor and Parry[51]). Condition (ND) means that $u_+(t)$ is nondecreasing. It is readily shown that $du/dt(t=0) < 0$ and $u_+(t)$ nondecreasing imply that $du/dt(t) < 0$ for all $t \geq 0$ (see pp. 526-527 of Taylor and Parry[51]). Consequently, from (35) we

TABLE I. Assumptions about Attrition-Rate Coefficients.

CONDITION (A): $\int_{t_0}^{t} a(s)ds$ and $\int_{t_0}^{t} b(s)ds$ are bounded

for all finite $t \geq t_0$.

CONDITION (B): $\lim_{t\to+\infty} \int_{0}^{t} b(s)ds = +\infty$.

CONDITION (C): $\lim_{t\to+\infty} \int_{t_0}^{t} \sqrt{a(s)b(s)}\, ds = +\infty$.

CONDITION (ND): $R(t)$ is nondecreasing on $[0,+\infty)$.

see that $x_0/y_0 < \sqrt{a_0/b_0}$ and Condition (ND) yield that $du/dt(t) < 0 \ \forall \ t \geq 0$. It then remains to be shown that X's breakpoint force ratio is reached in finite time (see Note 15). Let us observe that $R(t)$ nondecreasing and $0 < a(t), b(t) < +\infty \ \forall \ t \ (0,+\infty)$ $\Rightarrow$ $a_0 < +\infty$ and $b_0 > 0$. From Condition (ND) and the fact that $du/dt(t) < 0 \forall \ t \geq 0$, we have

$$\frac{du}{dt}(t) = b(t)\{u^2 - R(t)\} \leq \frac{b(t)}{b_0}\{b_0 u_0^2 - a_0\} = \frac{b(t)}{b_0}\frac{du}{dt}(t=0).$$

Hence

$$u(t) = u_0 + \int_0^t \left(\frac{du}{dt}\right)dt \leq u_0 + \frac{1}{b_0}\frac{du}{dt}(t=0)\int_0^t b(s)ds,$$

so that $\lim_{t\to+\infty} \int_0^t b(s)ds = +\infty$ implies that $u(t)$ goes to $u_Y^f \geq 0$ in finite time.

Q.E.D.

We observe that Theorem 2 says that the X force will be annihilated in finite time (i.e. $u(t=t_a^X) = 0$ for t_a^X finite when $u_Y^f = 0$).

We will now deduce some new results that many times allow us to predict such a battle's outcome from the initial conditions. Let $U = -(1/X)dX/d\tau$ so that (20) becomes the Riccati equation

$$\frac{dU}{d\tau} = U^2 - 1 - F(\tau) \qquad \text{with} \qquad U(\tau=\tau_0) = \frac{y_0}{x_0}\sqrt{\frac{a_0}{b_0}} + \varepsilon_0. \quad (36)$$

We observe that

$$U(\tau) = \frac{y}{x}R^{1/2} + \varepsilon(\tau). \quad (37)$$

If $F(\tau) \leq 0 \ \forall \ \tau \geq \tau_0$, then $dU/d\tau \geq U^2 - 1$, and $U(\tau=\tau_0) = U_0 > 1 \Rightarrow \lim_{\tau\to\tau_{X_a}} U(\tau) = +\infty$ for τ_{X_a} finite. We observe from (22) and the definition of t_0 that $R(t(\tau))$ and $\varepsilon(\tau)$ are finite for all $\tau \geq \tau_0 > 0$ so that by (37) the X force must be annihilated in finite time, since by Conditions (A) and (C) $\tau(t)$ is a strictly increasing function $\forall \ t \in [t_0,+\infty)$ with range $[0,+\infty)$. If we additionally assume that $dR/dt \leq 0$, then $du/dt(t) < 0$ for all $t \in [0,t_X^a]$ with $\lim_{t\to t_{X_a}} u(t) = 0$ so that the X force will

lose any fixed-force-ratio-breakpoint battle in finite time. The proof is by contradiction: $dR/dt \leq 0 \Rightarrow u_+(t)$ nonincreasing $\Rightarrow$ if there exists t_1 such that $dx/dt(t_1) \geq 0$, then $du/dt(t) \geq 0 \; \forall \; t \geq t_1 \Rightarrow$ impossible to have $u \to 0$. Thus, we have proved

THEOREM 3: Assume that Conditions (A) and (C) hold. If $F(\tau) \leq 0 \; \forall \; \tau \geq \tau_0$, then $\frac{x_0}{y_0}(1-\varepsilon_0) < \sqrt{\frac{a_0}{b_0}}$ implies that the X force will be annihilated in finite time. Furthermore, if $dR/dt \leq 0 \; \forall \; \tau \geq \tau_0$, then the X force will lose any fixed-force-ratio-breakpoint battle in finite time.

If $F(\tau) \geq 0$ and $dR/dt \geq 0 \; \forall \; \tau \geq \tau_0$, then $dU/d\tau \leq U^2 - 1$ and $U_0 < 1 \Rightarrow \exists \; \tau_Y^a$ finite such that $U(\tau_Y^a) = 0$ with $U(\tau) > 0 \; \forall \; \tau \in [\tau_0, \tau_Y^a)$. Since $dR/dt \geq 0 \Rightarrow \varepsilon(t) \geq 0$, we see from (37) that Y must be annihilated in finite time, since $\tau(t)$ is a one-to-one mapping of $[t_0, +\infty)$ onto $[0, +\infty)$. By observing that

$$\frac{du}{dt} = b(t)u^2\{\frac{1}{2a(t)u}\frac{dR}{dt} + 1 - U^2 + (\frac{d}{d\tau}\ell n\, P)^2\},$$

we see that $du/dt(t) > 0 \; \forall \; t \geq 0$ as long as $u > 0$. Hence $u(t)$ is strictly increasing for $t \in [0, t_Y^a)$ and $\lim_{t \to t_{Y_a}^-} u(t) = +\infty$ for t_Y^a finite so that Y will lose any battle with $u_X^f > u_0$. Thus, we have proved

THEOREM 4: Assume that Conditions (A), (B), and (ND) hold. If $F(\tau) \geq 0 \; \forall \; \tau \geq \tau_0$, then $\frac{x_0}{y_0}(1-\varepsilon_0) > \sqrt{\frac{a_0}{b_0}}$ implies that the Y force will lose a fixed-force-ratio-breakpoint battle in finite time.

By considering Lioville's normal form (24) for the Y force-level equation, one may similarly prove Theorems 5 and 6.

THEOREM 5: Assume that Conditions (A) and (C) hold. If $G(\tau) \leq 0 \; \forall \; \tau \geq \tau_0$, then $\frac{x_0}{y_0} > \sqrt{\frac{a_0}{b_0}}(1+\varepsilon_0)$ implies that the Y force will be annihilated in finite time. Furthermore, if $dR/dt \geq 0 \; \forall \; \tau \geq \tau_0$ (i.e. Condition (ND) holds), then Y will lose any fixed-force-ratio-breakpoint battle in finite time.

THEOREM 6: Assume that $\lim_{t\to+\infty} \int_0^t a(s)ds = +\infty$ and that Condition (A) holds. If $G(\tau) \geq 0$ and $dR/dt \leq 0 \; \forall \; \tau \geq \tau_0$, then $\frac{x_0}{y_0} < \sqrt{\frac{a_0}{b_0}}\,(1+\varepsilon_0)$ implies that the X force will lose a fixed-force-ratio-breakpoint battle in finite time.

Although $F(\tau)$ and $G(\tau)$ are related by (29), for all the attrition-rate coefficients that we have so far considered, $F(\tau) \geq 0 \Leftrightarrow G(\tau) \leq 0$. Also, in all these cases $F(\tau) \geq 0 \Leftrightarrow dR/dt \geq 0$. Since $x_0/y_0 < \sqrt{a_0/b_0}\,(1+\varepsilon_0) \Rightarrow (x_0/y_0)(1-\varepsilon_0) < \sqrt{a_0/b_0}$ and $(x_0/y_0)(1-\varepsilon_0) > \sqrt{a_0/b_0} \Rightarrow x_0/y_0 > \sqrt{a_0/b_0}\,(1+\varepsilon_0)$, we see that Theorem 6 is stronger than Theorem 3 and Theorem 4 stronger than Theorem 5. Under these conditions, Theorems 4 and 6 provide no additional information on battle outcome over that contained in Theorems 3 and 5.

7. Application to General Power Attrition-Rate Coefficients.

We now apply the above results to combat between two homogeneous forces modelled by (7) with the general power attrition-rate coefficients (8). We distinguish between two cases: (I) power attrition-rate coefficients with no offset (i.e. $A = 0$), and (II) power attrition-rate coefficients with the same parity (i.e. $\mu = \nu$) and positive offset. In order to invoke Theorems 3 through 6, we must have $C > 0$ for the general power attrition-rate coefficients (8) (cf. (19)). In order that Condition (A) be satisfied, we must have $\mu,\nu > -1$; and then Conditions (B) and (C) hold (see Table I).

7.1. Power Attrition-Rate Coefficients with No Offset.

In this case we have

$$a(t) = k_a(t+C)^{\mu} \qquad \text{and} \qquad b(t) = k_b(t+C)^{\nu}, \tag{38}$$

so that

$$dR/dt \geq 0 \;\Leftrightarrow\; \mu \geq \nu. \tag{39}$$

Theorem 2 then yields

COROLLARY 2.1: For the power attrition-rate coefficients (38) with $\mu \geq \nu$ and $C > 0$, $\frac{x_0}{y_0} < \sqrt{\frac{a_0}{b_0}}$ implies that the Y force will win a fixed-force-ratio-

breakpoint battle in finite time. In particular, the X force will be annihilated in finite time when $u_Y^f = 0$.

In preparation for invoking Theorems 3 through 6, we compute

$$F(\tau) = \frac{(\mu-\nu)(3\mu+\nu+4)}{4(\mu+\nu+2)^2\tau^2}, \tag{40}$$

and

$$G(\tau) = \frac{(\nu-\mu)(\mu+3\nu+4)}{4(\mu+\nu+2)^2\tau^2}, \tag{41}$$

since (13) with (38) yields

$$\tau = \tau(t) = \frac{(2\sqrt{k_a k_b})}{(\mu+\nu+2)}(t+C)^{(\mu+\nu+2)/2}, \tag{42}$$

with $\tau_0 = (2\sqrt{k_a k_b}/(\mu+\nu+2))C^{(\mu+\nu+2)/2} > 0$. We observe that

$$F(\tau) \geq 0 \quad \text{and} \quad G(\tau) \leq 0 \; \forall \; \tau > 0 \quad \Leftrightarrow \quad \mu \geq \nu. \tag{43}$$

Thus, as discussed above at the end of Section 6, Theorems 4 and 6 provide no additional information on force annihilation over that contained in Theorems 3 and 5. Furthermore, Theorem 3 is stronger than Theorem 2, since ε_0, shich is given by (22) evaluated at $t = 0$, is ≥ 0. Hence, we omit any corollary to Theorem 3. As a corollary to Theorem 5 we have

COROLLARY 5.1: For the power attrition-rate coefficients (38) with $\mu \geq \nu$ and $C > 0$, $\frac{x_0}{y_0} > \sqrt{\frac{a_0}{b_0}}\{1+\frac{(\mu-\nu)}{4\sqrt{k_a k_b}}C^{-(\mu+\nu+2)/2}\}$ implies that the X force will win a fixed-force-ratio-breakpoint battle in finite time. In particular, the Y force will be annihilated in finite time when $u_X^f = +\infty$.

Let us also write the X-victory-prediction condition of Corollary 5.1 as

$$\frac{x_0}{y_0} > \sqrt{\frac{k_a}{k_b}}C^{(\mu-\nu)/2} + \frac{(\mu-\nu)}{4k_b}C^{-(\nu+1)} = f(C). \tag{44}$$

Observing that $\lim_{C\to 0^+} f(C) = +\infty$ for $\mu > \nu$, we see that the victory-prediction conditions of Corollaries 2.1 and 5.1 become stronger as C decreases and are meaningless for $C = 0$.

We now show that other considerations, however, readily yield force-annihilation conditions when $C = 0$. [Unfortunately, without further analysis these results are limited to force-annihilation prediction and do not apply directly to fixed-force-ratio-breakpoint battles with $u_X^f < +\infty$ and $u_Y^f > 0$.] The solution to (7) with the power attrition-rate coefficients (38) may be written for $C = 0$ as (see Note 16)

$$x(t) = (p^p/(2p))\{C_X+C_Y\}A_\beta(T) + (p^p/(2\sqrt{p}))\{C_X-C_Y\}B_\beta(T), \tag{45}$$

where A_β and B_β denote the generalized Airy functions of the first and second kinds of order β (see SWANSON and HEADLEY[41]), $p = (\mu+1)/(\mu+\nu+2)$, $\beta = (\nu-\mu)/(\mu+1)$, $C_X = x_0\,\Gamma(1-p)$, $C_Y = y_0\sqrt{k_a/k_b}\,\Gamma(p)(\sqrt{k_a k_b}/(\mu+\nu+2))^{1-2p}$, and $T = (\sqrt{k_a k_b}/(\mu+1))^{2p} t^{\mu+1}$. Observing that [41] $A_\nu(\xi)$, $B_\nu(\xi) > 0\ \forall\ \xi \geq 0$, $\lim_{\xi\to+\infty} A_\nu(\xi) = 0$, and $\lim_{\xi\to+\infty} B_\nu(\xi) = +\infty$, we see from (45) that $\lim_{t\to+\infty} x(t) = -\infty$ if and only if the coefficient of $B_\beta(T)$ is negative (i.e. $C_X < C_Y$). Hence, we may conclude (cf. the development of (3) from (5))

PROPOSITION 1: For a fight-to-the-finish modelled with the power attrition-rate coefficients (38) and $C = 0$, the X force will be annihilated in finite time if and only if $\dfrac{x_0}{y_0} < \sqrt{\dfrac{k_a}{k_b}}\left(\dfrac{\sqrt{k_a k_b}}{\mu+\nu+2}\right)^{1-2p}\dfrac{\Gamma(p)}{\Gamma(1-p)}$.

For $C > 0$, (45) does not take nearly such a convenient form (see Taylor and Brown[50]), and the analogue of Proposition 1 for $C > 0$ would involve the generalized Airy functions and their derivatives evaluated at a positive argument. Thus, the prediction of force annihilation by this approach for $C > 0$ would require tabulations of higher transcendental functions. Such tabulations do not currently exist (see reference 50). Thus, we see the usefulness of the "strong" annihilation conditions given by Corollaries 2.1 and 5.1, which contain only elementary functions: "exact" annihilation-prediction conditions involve higher transcendental functions.

7.2. Power Attrition-Rate Coefficients with the Same Parity and Positive Offset.

In this case we have

$$a(t) = k_a(t+C)^\mu \qquad \text{and} \qquad b(t) = k_b(t+C+A)^\mu, \tag{46}$$

with $A, C > 0$. It follows that

$$dR/dt \gtrless 0 \quad \Leftrightarrow \quad \mu \gtrless 0. \tag{47}$$

Then Theorem 2 yields

COROLLARY 2.2: For the offset power attrition-rate coefficients (46) with $\mu \geq 0$ and $A,C > 0$, $\frac{x_0}{y_0} < \sqrt{\frac{a_0}{b_0}}$ implies that the Y force will win a fixed-force-ratio-breakpoint battle in finite time.

Let us now see what information is yielded by Theorems 3 through 6. It is more convenient to express F and G in terms of the old time variable t (see (27) and (28)). Then

$$F(\tau) = \frac{\mu A\{4(\mu+2)(t+C) + (3\mu+4)A\}}{16k_a k_b (t+C)^{\mu+2}(t+C+A)^{\mu+2}}, \tag{48}$$

and

$$G(\tau) = \frac{-\mu A\{4(\mu+2)(t+C) + (\mu+4)A\}}{16k_a k_b (t+C)^{\mu+2}(t+C+A)^{\mu+2}}. \tag{49}$$

We observe that

$$F(\tau) \geq 0 \quad \text{and} \quad G(\tau) \leq 0 \; \forall \, \tau > 0 \quad \Leftrightarrow \quad \mu \geq 0, \tag{50}$$

so that Theorems 4 and 6 again provide no additional information over that contained in Theorems 3 and 5. Furthermore, we may omit consideration of Theorem 3, since it is implied by Theorem 2. As a corollary to Theorem 5 we have

COROLLARY 5.2: For the offset power attrition-rate coefficients (46) with $\mu \geq 0$ and $A,C > 0$, $\frac{x_0}{y_0} > \sqrt{\frac{a_0}{b_0}}\left\{1 + \frac{\mu A}{4\sqrt{a_0 b_0}\, C(C+A)}\right\}$ implies that the X force will win a fixed-force-ratio-breakpoint battle in finite time.

Let us also write the X-victory-prediction condition of Corollary 5.2 as

$$\frac{x_0}{y_0} > \sqrt{\frac{k_a}{k_b}}\,(1+A/C)^{-\mu/2} + \frac{\mu A}{4b_0 C(C+A)} = g(C). \tag{51}$$

We observe that $\lim_{C \to 0^+} g(C) = +\infty$, and we again see that the victory-prediction conditions of Corollaries 2.2 and 5.2 become stronger as C decreases and become meaningless for $C = 0$.

8. Discussion.

In his classic 1914 paper [26], Lanchester assumed that the combatants' fire effectivenesses (as expressed by the Lanchester attrition-rate coefficients) were constant over time and deduced his famous square law (2), which allows one to tradeoff quality versus quantity of weapon systems by means of the condition for equality of "fighting strengths"

$$x_0/y_0 = \sqrt{a/b}, \tag{52}$$

where a and b denote constant attrition-rate coefficients (see Section 2). Thus, we see that equality of Lanchester-type fighting strengths (see Note 17) depends on two parameters: the initial force ratio and the relative effectiveness. When the timing of military actions is considered, we add a third parameter, the intensity of combat, to this list of significant combat parameters. In the paper at hand, we extended these well-known constant-coefficient results to battles between two homogeneous forces with temporal variations in the fire effectivenesses.

No such simple relationship like the square law (2), which yielded (52), holds in general for variable attrition-rate coefficients (see Note 18). By transforming the independent variable to normalize the battle's time scale by the intensity of combat, we found that the course of combat depends on two weapon system parameters: (I) relative fire effectiveness, $R(t) = a(t)/b(t)$, and (II) intensity of combat, $I(t) = \sqrt{a(t)b(t)}$. Moreover, when the temporal variations in relative fire effectiveness $R(t)$ follow a regular pattern (e.g. $R(t)$ nondecreasing), the battle's outcome can many times be predicted from the battle's initial conditions. To obtain such results, we considered Liouville's normal form for the, for example, X force-level equation and found that it not only yields new battle-outcome-prediction conditions but also suggests an approximation to the time history of the X force level $x(t)$. As seen from (21), Liouville's normal form (20) introduces second order conditions (e.g. the second derivative of relative fire effectiveness with respect to transformed time) into Lanchester-type combat analysis.

The new victory-prediction results (see Theorems 3 through 6) that we have developed here are complementary to those of Taylor and Parry[51] (see Theorem 2). This complementary nature is seen by observing that under the appropriate comparable conditions we have

from Taylor and Parry's Theorem 2 $(\frac{dR}{dt} \geq 0)$: Y will win if $\frac{x_0}{y_0} < \sqrt{\frac{a_0}{b_0}}$,

from our new results, Theorem 5 $(G(\tau) \leq 0)$: X will win if $\frac{x_0}{y_0} > \sqrt{\frac{a_0}{b_0}}(1+\varepsilon_0)$,

where

$$\varepsilon_0 = \frac{1}{\sqrt{a_0 b_0}} \{ \frac{d}{dt} \ell n \left[\frac{a(t)}{b(t)} \right]^{1/4} \}_{t=0} . \tag{53}$$

For both special cases of general power attrition-rate coefficients considered in Section 7 above we had $dR/dt \geq 0 \Leftrightarrow F(\tau) \geq 0 \Leftrightarrow G(\tau) \leq 0 \Leftrightarrow \varepsilon_0 \geq 0$. Although these if-and-only-if statements do not hold in general, they do hold for these particular coefficients. In both cases, we observe that for

$$\sqrt{\frac{a_0}{b_0}} \leq \frac{x_0}{y_0} \leq \sqrt{\frac{a_0}{b_0}} (1+\varepsilon_0), \tag{54}$$

we cannot say by this approach who will be the loser of the fixed-force-ratio-breakpoint battle (see Figure 3). We observe that force annihilation is a special case of these outcome-prediction conditions. From both (54) and Figure 3, we see that there is a "gap" in these victory-prediction conditions (i.e. Theorems 2 through 6). The price of removing this "gap," however, is the introduction of higher transcendental functions (see Section 7 above and Taylor and Brown[50]). Furthermore, "exact" results with no such gap in the victory-prediction conditions are apparently only possible for a fight-to-the-finish in which one side or the other is to be annihilated.

We also refined a victory-prediction result of Taylor and Parry[51] (as applied to the model (7) under study) by adding a restriction on the attrition-rate-coefficients, i.e. our Condition (B), to the assumptions of Theorem 2. Taylor and Parry did not note that such a condition must be assumed in order to insure that the battle will

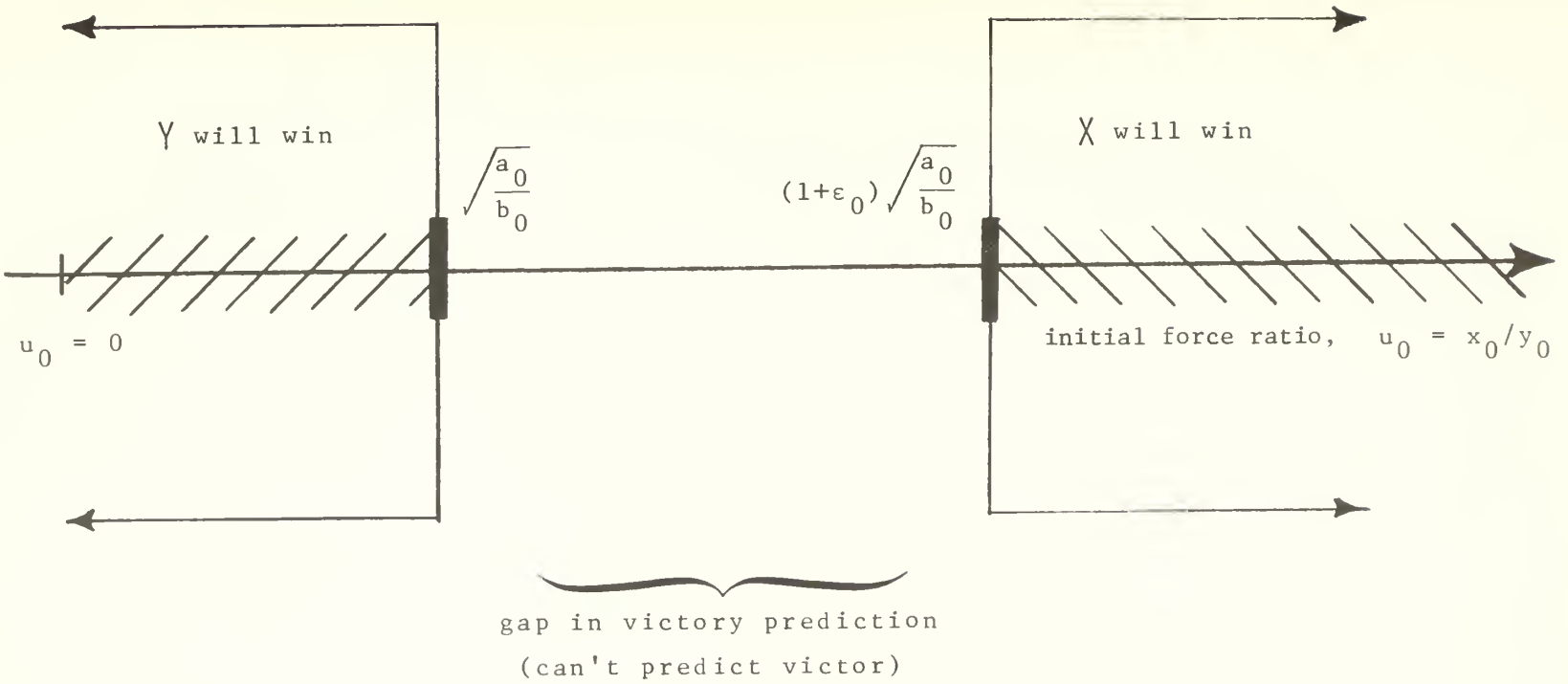

Figure 3. "Gap" in the Victory-Prediction Conditions for $dR/dt \geq 0$, $F(\tau) \geq 0$, $G(\tau) \leq 0$, $\varepsilon_0 > 0$.

terminate. All our new results, i.e. Theorems 3 through 6, contain some such restriction on the attrition-rate coefficients.

The results of this paper may be used in parametric analyses (see BONDER[7]) of the dynamic combat interactions between two homogeneous forces with time-(or range-) dependent weapon system capabilities. Such models are of particular interest in light of the work of S. BONDER[4,6] and others[1,8,25,37,38] on the prediction of Lanchester attrition-rate coefficients from weapon system performance data and the work of G. CLARK[12] on the estimation of such (time-dependent) coefficients from Monte Carlo simulation output. A further discussion of applications is to found in references 8 and 43. As is always the case, however, the insights gained into combat dynamics from such Lanchester-type models are no more valid than the models themselves.

9. Summary.

In this paper we have developed insights into the dynamics of Lanchester-type combat between two homogeneous forces with temporal variations in weapon system effectivenesses by considering Liouville's normal form for the X and Y force-level equations. Our principal results were some new outcome-prediction results that complemented those of Taylor and Parry[51]. We also saw that Taylor and Parry's victory-prediction result (as applied to our model (7)), Theorem 2, must be refined by making an additional assumption about the attrition-rate coefficients. Liouville's normal form also suggests an approximation, the Liouville-Green-Lanchester approximation, to the force-level trajectories. Additionally, by transforming the battle's time scale, we saw that the relative fire effectiveness of the two combatants and the intensity of combat were two key parameters affecting the course of battle. Our new victory-prediction conditions for fixed-force-ratio-breakpoint battles were applied to two special cases of combat modelled with general power attrition-rate coefficients: (I) power attrition-rate coefficients with no "offset" (modelling, for example, two weapon systems with the same miximum effective range), and (II) offset power attrition-rate coefficients with the same parity (modelling, for example, weapon systems with different

maximum effective ranges but the same type of range dependence). We saw that there was a "gap" in the range of initial force ratios for which we could predict the outcome of such a fixed-force-ratio-breakpoint battle. The price that one has to pay to remove this "gap" was discussed.

NOTES

1. F. W. Lanchester (1868-1946) was an English automotive and aeronautical engineer. For a brief sketch of his many scientific and engineering contributions, see McCLOSKEY [29]. In acknowledgment of his contribution to operations research (again, see reference 29) the Operations Research Society of American annually awards the Lanchester Prize (see p. 113 of reference 34) "for the paper on operations research judged to be the best of the calendar year."

2. See, for example, Bonder and Farrell[8], Bonder and Honig[9], Brackney[11], DEITCHMAN[14], MORSE and KIMBALL[30], SCHAFFER[39], Taylor and Parry[51], WALLIS[52], and Weiss[54,55].

3. As work by Bonder and Farrell[8], Taylor[43], and Taylor and Brown[50] shows, the infinite-series solution to variable-coefficient equations by itself provides little information about battle outcome because of its complexity.

4. In his well-known survey paper on the Lanchester theory of combat, Dolansky[15] suggested the development of outcome predicting relations without solving in detail and/or computing force-level trajectories as one of several problems for future research. The work at hand is a step towards this problem's resolution (see also Taylor and Parry[51] and Taylor[48]).

5. Scientific verification of Lanchester-type models (as with any combat model) is still an unresolved question (see Bonder[7]). Although there have been numerous attempt to compare the theoretical implications of such models (invariable quite simple, constant-coefficient ones) with empirical (i.e. historical) evidence (see, for example, ENGEL[16], HELMBOLD[19,20], SCHMIEMAN[40], WEISS[54,57], and WILLARD[58]), the results have, unfortunately, been inconclusive, with far from universal agreement as to their correct interpretation. The historical data base is apparently not rich enough in detail to permit a definitive answer to the scientific validity of Lanchester-type models (see HELMBOLD[21]), since nations fight wars for other reasons than to collect combat data.

6. H. K. WEISS[56] has pointed out that Lanchester, an Englishman, was anticipated (in qualitative but not quantitative terms) in 1905 by Bradley A. Fiske (then Commander but later Rear Admiral, USN), an American. For a sketch of the life and accomplishments of Bradley Allen Fiske (1854-1942), see pp. 298-299 of reference 31. J. ENGEL[17] subsequently showed that Fiske's verbal model is equivalent to a system of difference equations (in contrast to Lanchester's differential equations) and examined some of the mathematical consequences of these Fiske-type equations of warfare.

7. The equations (1) are only valid for $x,y > 0$. The first, for example, becomes $dx/dt = 0$ for $x = 0$. Moreover, there is far from universal agreement as to which variables are significant and can be used to predict the outcome of the combat process For some other views, see HAYWARD[18] and LIDDELL HART[27].

8. The influential 19th-century German military philosopher, Carl von Clausewitz (1780-1831), stated in his classic work On War (Vom Kriege) (see p. 276 of reference 13), "The best Strategy is always to be very strong, first generally then at the decisive point. ...There is no more imperative and no simpler law for Strategy than to keep the forces concentrated."

9. As pointed out by Taylor and Parry[51], the entire subject of modelling battle termination is a problem area in contemporary defense planning studies. There is far from universal agreement on this topic (see TAYLOR[47] for further references).

10. Before the mid-1960's the use of Lanchester-type models in defense planning studies was hampered by the inability to predict the attrition-rate coefficients (see Bonder[4]). Thus, two significant accomplishments for the Lanchester theory of combat in the 1960's were (I) the development of methodology for the prediction of Lanchester attrition-rate coefficients from weapon system performance data by S. Bonder[4,6] and others (see BARFOOT[1], Bonder and Farrell[8], and KIMBLETON[25]), and (II) G. Clark's[12] development of methodology for the (maximum likelihood) estimation of such coefficients from Monte Carlo simulation output. Both these developments and others (see references 10, 37, and 38) have facilitated the application in defense planning studies of models such as (7) and its generalization to combat between heterogeneous forces (see reference 8).

11. The modelling roles of A and C are discussed in Taylor and Brown[50].

12. Taylor and Brown[50] give an example of nonexistence of a solution to (7) for the power attrition-rate coefficients (8) with $A = 0$ (see Section 4 of reference 50).

13. The term quasi-autonomous was coined by Taylor[48] (see also TAYLOR[49]) to denote a system of differential equations transformable to an autonomous system (see, for example, p. 163 of PETROVSKI[35]) by a change of the time scale. Special cases of such Lanchester-type equations have been considered by, for example, Farrell[8] and TAYLOR[42]. More general (possibly nonlinear) quasi-autonomous Lanchester-type equations have been studied by Taylor[48,49] (see also Note 4 of Taylor and Brown[50]).

14. A similar theorem for a more general model with supporting fires was given by Taylor and Parry[51], but they did not observe in reference 51 that certain additional conditions must be assumed to insure that the battle will terminate. In other words, an analogous result originally given by Taylor and Parry[51] for such a linear, variable-coefficient model with supporting fires is not true in general without certain restrictions on attrition-rate coefficients being added.

15. This point was overlooked by Taylor and Parry[51] (see Note 14 above).

16. The substitution $s = K\sqrt{k_b/k_a}\int_0^t a(\sigma)d\sigma$ where $K = (\sqrt{k_a k_b}/(\mu+1))^{2p-1}$ transforms the X force-level equation (11) with power attrition-rate coefficients (38) and $C = 0$ into $d^2x/dx^2 - s^{\beta}x = 0$, which is readily recognized as the generalized Airy equation (see Swanson and Headley[41]).

17. The determination of equality of fighting strengths will be affected by the battle termination model used in the operational definition of the concept of fighting strength. For a discussion of related matters, see reference 18 in which P. Hayward examines the factors that determine combat effectiveness.

18. As apparently first observed by B. O. Koopman[30], a "square-law" relationship holds for the quasi-autonomous case in which $a(t)/b(t) = \text{constant}$ (see also reference 50).

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